ELECTROMAGNETIC FIELD THEORY

CONTENTS
(As per GGSIPU Syallabus)

- Introduction to Electromagnetic Theory.
- The electromagnetic field.
- Electromagnetic waves
- The Wave Equation
- Energy carried by the Electromagnetic Waves
- Poynting's Theorem
- Intensity of the Electromagnetic Wave
- Radiation Pressure (Momentum, Angular Momentum in em waves)

THE LIGHT WAVE

CONSTITUENT OF THE LIGHT

- THE ELECTRIC WAVE VECTOR

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0}\exp(i(\boldsymbol{k}.\boldsymbol{r} - \omega t))$$

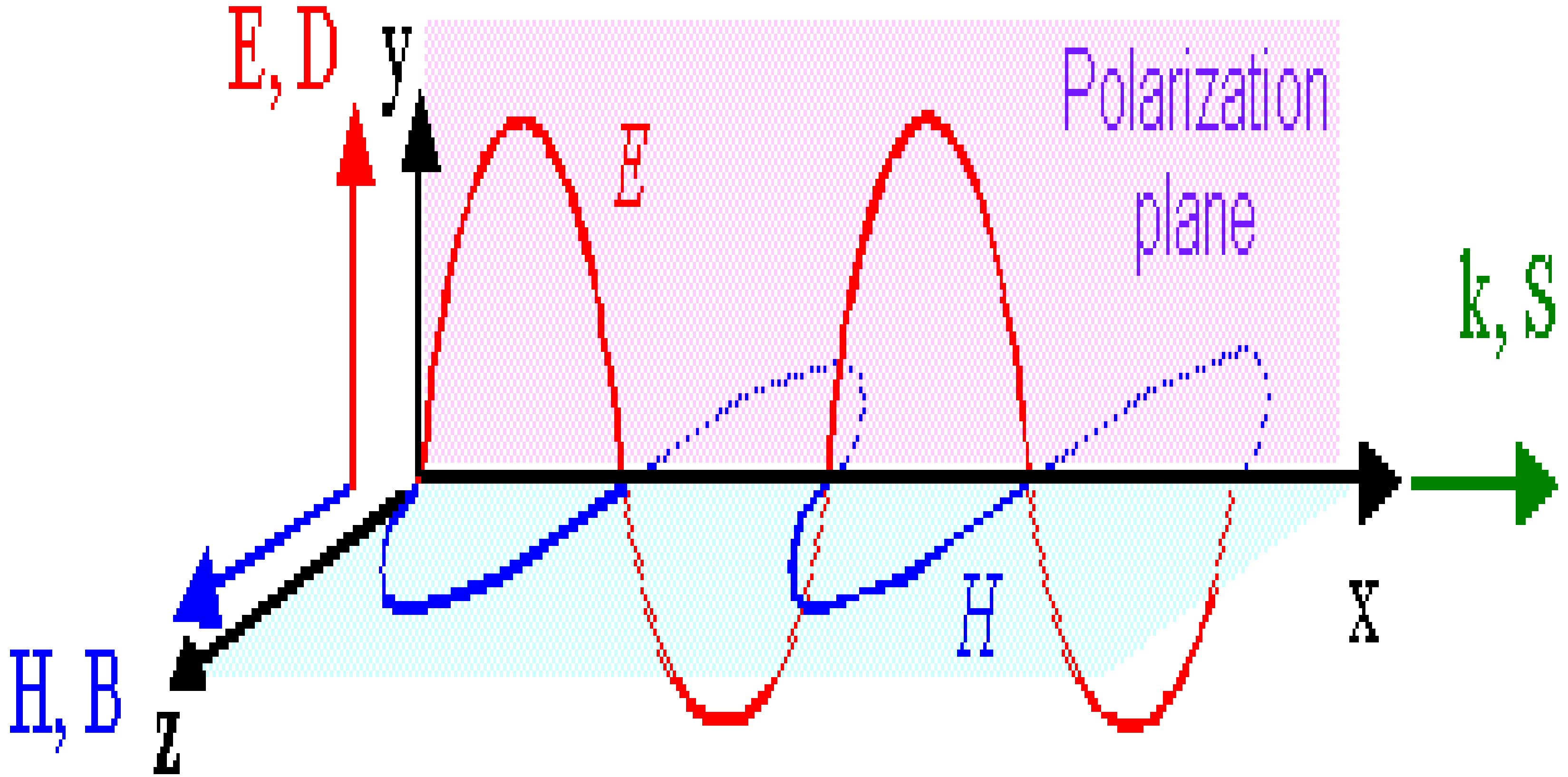
- THE MAGNETIC WAVE VECTOR

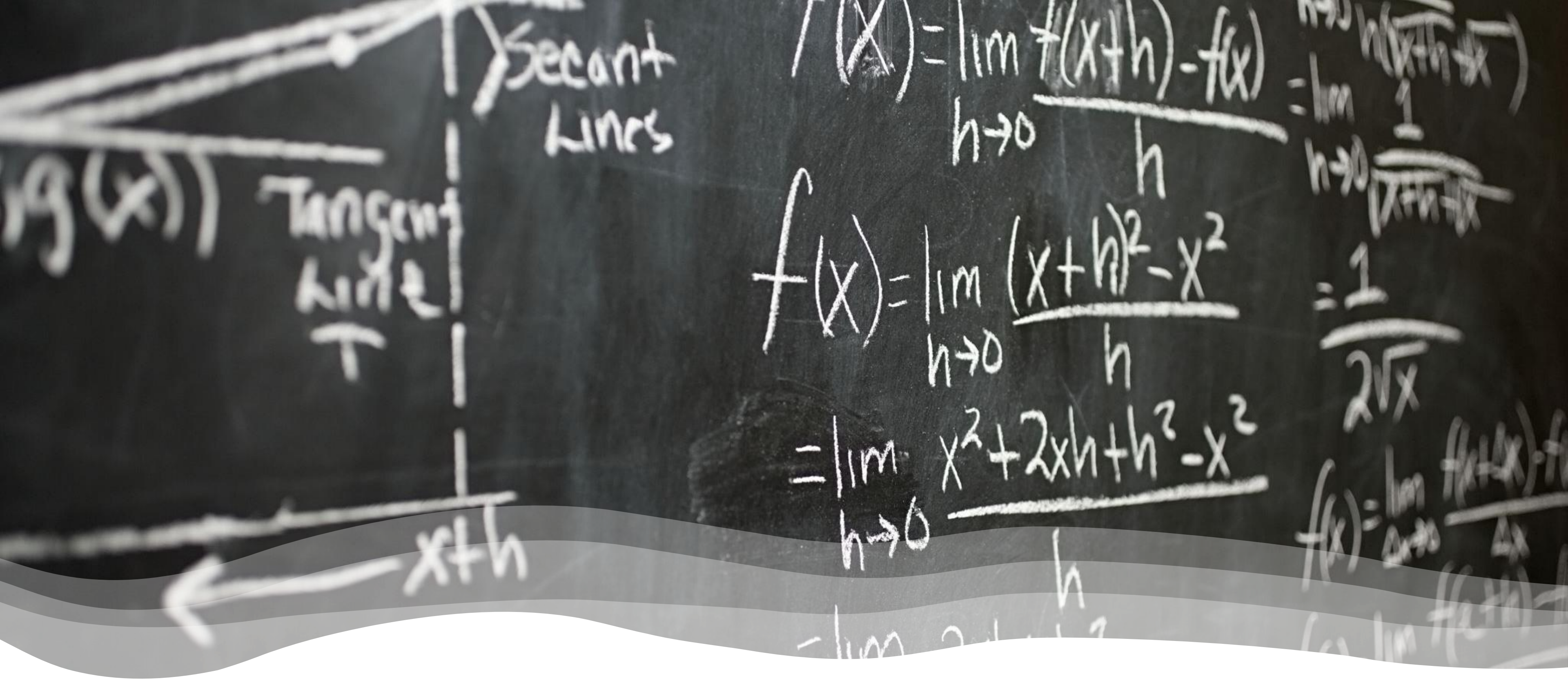
$$\mathbf{H}(\boldsymbol{r},t) = \boldsymbol{H_0}\exp\left(i(\boldsymbol{k}.\boldsymbol{r} - \omega t)\right)$$

Or

$$\mathbf{B}(\boldsymbol{r},t) = \boldsymbol{B_0}\exp\left(i(\boldsymbol{k}.\boldsymbol{r} - \omega t)\right)$$

GRAPHICAL REPRESENTATION OF ELECTRIC and MAGNETIC VECTOR





THE EM WAVE EQUATION HAS BEEN THE CONTRIBUTION OF MAXWELL

ELECTROMAGNETIC WAVE EQUATION

DIFFERENTIAL / POINT FORM

- 1. $\vec{\nabla}.\vec{D} = \rho$ GAUSS LAW IN ELECTROSTATICS
- 2. $\vec{\nabla}.\vec{B} = 0$ GAUSS LAW IN MAGNETOSTATICS
- 3. $\vec{\nabla} x \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Faraday's law of electromagnetic Induction

- 4. $\vec{\nabla} x \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

Ampere-Maxwell Circuital Law.

Magnetic flux / Electric flux

INTEGRAL FORM OF MAXWELL'S EQUATION

- 1. $\iint_S \vec{D}.\vec{dS} = \iiint_V \rho dV$
- 2. $\iint_S \vec{B}.\vec{dS} = 0$
- **3.** $\oint \vec{E}.dl = \iint_S -\frac{\partial \vec{B}}{\partial t}.dS$
- **4.** $\oint \vec{H}.dl = \iint_S (\vec{J_C} + \frac{\partial \vec{D}}{\partial t}).dS$

THE EM WAVE EQUATION

- Some, simple mathematical operations help us to derive the EM wave equation as done in the steps that follow.
- From the third Maxwell's equation we have:

$$\boldsymbol{\nabla}\ \boldsymbol{x}\ (\boldsymbol{\nabla}\ \boldsymbol{x}\ \boldsymbol{E}) = -\mu\ \frac{\partial}{\partial t}(\boldsymbol{\nabla x H})$$

$$\boldsymbol{\nabla}\ \boldsymbol{x}\ (\boldsymbol{\nabla}\ \boldsymbol{x}\ \boldsymbol{E}) = -\mu\ \frac{\partial}{\partial t}\left(\sigma\boldsymbol{E} + \epsilon\frac{\partial\boldsymbol{E}}{\partial t}\right)$$

Using the vector identity one gets:

$$\boldsymbol{\nabla}\ \boldsymbol{x}\ (\boldsymbol{\nabla}\ \boldsymbol{x}\ \boldsymbol{E}) = \boldsymbol{grad\ div\ E} - \boldsymbol{\nabla}^2\boldsymbol{E}$$

Thus, one gets,

$$\boldsymbol{grad\ div\ E} - \boldsymbol{\nabla}^2\boldsymbol{E} = -\sigma\mu\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{t}} - \epsilon\mu\frac{\partial^2\boldsymbol{E}}{\partial\boldsymbol{t}^2}$$

$$-\boldsymbol{\nabla}^2\boldsymbol{E} = -\sigma\mu\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{t}} - \epsilon\mu\frac{\partial^2\boldsymbol{E}}{\partial\boldsymbol{t}^2}$$

$$\boldsymbol{\nabla}^2\boldsymbol{E} - \sigma\mu\frac{\partial\boldsymbol{E}}{\partial\boldsymbol{t}} - \epsilon\mu\frac{\partial^2\boldsymbol{E}}{\partial\boldsymbol{t}^2} = \boldsymbol{0}$$

This is the *wave equation* for the electric wave vector.

THE EM WAVE EQUATION

- From the third Maxwell's equation we have:

$$\nabla\ x\ (\nabla\ x\ H) = -\epsilon\ \frac{\partial}{\partial t}(\nabla x E)$$

$$\nabla\ x\ (\nabla\ x\ H) = -\epsilon\ \mu \frac{\partial H}{\partial t}$$

Using the vector identity one gets:

$$\nabla\ x\ (\nabla\ x\ H) = grad\ div\ H - \nabla^2 H$$

Thus, one gets,

$$grad\ div\ H - \nabla^2 H = -\sigma\mu\frac{\partial H}{\partial t} - \epsilon\mu\frac{\partial^2 H}{\partial t^2}$$

$$-\nabla^2 H = -\sigma\mu\frac{\partial H}{\partial t} - \epsilon\mu\frac{\partial^2 H}{\partial t^2}$$

$$\nabla^2 H - \sigma\mu\frac{\partial E}{\partial t} - \epsilon\mu\frac{\partial^2 E}{\partial t^2} = 0$$

This is the *wave equation* for the magnetic wave vector.

THE VELOCITY OF AN EM WAVE

- Thus, owing to the wave equation both the electric and the magnetic vector can be written in the following forms:
- $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E_0} \exp\big(i(\boldsymbol{k}.\boldsymbol{r} \ -\omega t)\big)$
- $\mathbf{H}(\boldsymbol{r},t) = \boldsymbol{H_0} \exp\big(i(\boldsymbol{k}.\boldsymbol{r} \ -\omega t)\big)$

Thus, it is evident from conventions that $\check{k}$, is the propagation wave vector, $\check{r}$ is the position vector, and ω is the angular .

- The angular velocity , ω, is directly proportional to the *speed of propagation :*

$$v = \frac{\omega}{\check{k}} = \frac{1}{\sqrt{\mu\epsilon}}$$

- *For a light wave this velocity has been verified to be the speed of light.*

DEFINING A LINEAR,ISOTROPIC and HOMOGENEOUS MEDIUM

The constitutive relations for a linear, homogeneous and isotropic medium are :

1. $\boldsymbol{D = \epsilon E}$
2. $\boldsymbol{B = \mu H}$
3. $\boldsymbol{J = \sigma E}$

where, E is the Electric Vector,

D is the Electric Displacement Vector

B is the Magnetic Inductionand

H is the Magnetic Field

Also, ϵ *is the Dielectric Permitivitty*

μ *is the Magnetic Permeability*

σ *is the Conductivity of the medium*

ELECTROMAGNETIC ENERGY INTENSITY

The electric and magnetic fields are associated with a certain flow of energy.

The two different forms of Energy that we know are the *Kinetic energy* and the *Potential Energy*

Thus, under the Maxwell's modified theory, both the electric and the magnetic energy together contribute towards the flow of energy in a medium.

This introduces the new concept of 'ELECTROMAGNETIC ENERGY DENSITY'

ELECTROMAGNETIC ENERGY
STATIC FIELDS

- The Electromagnetic Potential Energy is defined as :

$$U_c = \frac{1}{2} \iiint_V E.B\, dV$$

- The magnetic field is associated with an energy represented as :

$$U_m = \frac{1}{2} \iiint_V H.B\, dV$$

Both the above energies are associated with the *Static or the Time-Independent* situations.

ELECTROMAGNETIC ENERGY
NON-STATIC FIELDS

- We try to understand and derive the flow of energy for the *Non-Static fields* :

$$\boldsymbol{\nabla x E} = -\frac{\partial \boldsymbol{B}}{\partial t}\ ;\ \boldsymbol{\nabla x H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

- Taking the scalar products of H and E respectively,

$$\mathbf{H}.(\boldsymbol{\nabla x E}) = -\boldsymbol{H}\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\mathbf{E}.(\boldsymbol{\nabla x H}) = \boldsymbol{E}.\boldsymbol{J} + \boldsymbol{E}.\frac{\partial \boldsymbol{D}}{\partial t}$$

- Subtracting the two expressions:

$$\mathbf{H}.(\boldsymbol{\nabla x E}) - \mathbf{E}.(\boldsymbol{\nabla x H})$$

$$= -(\boldsymbol{H}\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E}.\frac{\partial \boldsymbol{D}}{\partial t}) - (\mathbf{E.J})$$

Using the vector identity :

$$\nabla.(\boldsymbol{AxB}) = \mathbf{A}.(\boldsymbol{\nabla x B}) - \mathbf{B}.(\boldsymbol{\nabla x A})$$

- ***Hence, one gets,***

$$\nabla.(\boldsymbol{ExH}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}(\boldsymbol{E}.\boldsymbol{D} + \boldsymbol{H}.\boldsymbol{B})\right) - (\mathbf{E.J})$$

- **Div (S) +** $\frac{\partial \boldsymbol{U}}{\partial t} = -\boldsymbol{J}.\boldsymbol{E}$

Where, $\boldsymbol{S} = (\boldsymbol{E\,xH})$

$\boldsymbol{S}$, ***is the POYNTING VECTOR***

WORK DONE BY THE EM FIELD

- The energy is represented as :

$$U = \frac{1}{2}\, B.H + \frac{1}{2}\, D.E$$

- If a charge 'q' is acted upon by an em field then the work done by the field in moving it through a distance **ds** would be **(F.ds), hence the work done per unit time will be:**

$$\boldsymbol{F.\frac{ds}{dt} = F.v = \left(qE + q(vx\,B)\right).v}$$

$$\boldsymbol{F = qE.v}$$

- If there are N charged particles per unit volume , each carrying a charge 'q', then the work done per unit volume would be :

$$N q\boldsymbol{v}.\boldsymbol{E} = \boldsymbol{J}.\boldsymbol{E}$$

Where J represents the current density, (J.E) is akin to the Joule Loss.

ELECTROMAGNETIC ENERGY DENSITY NON-STATIC FIELDS

- $$div\,(\boldsymbol{S}) + \frac{\partial \boldsymbol{U}}{\partial t} = -\boldsymbol{J}.\boldsymbol{E} \quad ,$$

where, $$\boldsymbol{U} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\,(\boldsymbol{E}.\boldsymbol{D} + \boldsymbol{H}.\boldsymbol{B})$$

represents the electrostatic and the magnetic energy respectively.

- Hence, there is continuity and conservation in the flow of the electromagnetic energy .
- The term on the right hand side **(-J.E)** represents the kinetic energy of the charged particles , similar to the Joule Loss.
- **In the integral form one can write:**

$$\oint \boldsymbol{S}.\boldsymbol{dS} + \int \boldsymbol{J}.\boldsymbol{E}\,\boldsymbol{dV} = -\frac{\boldsymbol{d}}{\boldsymbol{dt}} \int \boldsymbol{U}\,\boldsymbol{dV}$$

This is referred to as the POYNTING THEOREM .

Where, the LHS of the equation represents the sum of the net flow out of energy plus the Joule Loss for a given volume enclosing that surface. The RHS represents the rate of decrease of the total energy. This equation represents the Equation of Continuity or the Conservation of Electromagnetic Energy Density.

INTENSITY OF THE ELECTROMAGNETIC WAVE

- The intensity of an electromagnetic wave is the power transmitted per second by the wave that impinges on an area A.
- Intensity (by definition) is the average energy transfer.
- The intensity, can be thus mathematically, be represented as the product of the average of the radiation and the wave(radiation) velocity :

$$I = \frac{1}{2}\, \epsilon\, v\, E_0^2$$

- The velocity of the propagation of a wave is given as follows:

$$v = \frac{1}{\sqrt{\epsilon\mu}}$$

$$I = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}}\; E_0^2$$

- In free space,

$$I = \frac{1}{2}\; \epsilon_0 c\, E_0^2$$

Substituting the value of permittivity and the velocity of light wave, one gets:

$$I = \frac{1}{2}\left(8.854\, x \frac{10^{-12} C^2 m^2}{N}\right) x \left(\frac{3x10^8 m}{s}\right) E_0^2$$

$$I = \left(1.33x\; 10^{-3}\; \frac{W}{V^2}\right) E_0^2$$

POYNTING VECTOR

The *Poynting Vector* represents the directional energy flux density (the rate of energy transfer per unit area , in units of watts per square metre W/m^2) of an electromagnetic field.

$$S = (E \; x \; H)$$

This vector gives the direction of flow of an electromagnetic radiation.

The quantity **S.da is the electromagnetic energy crossing the area da per unit time. The radiant flux density is the average value of the Poynting vector over a given time interval. When energy is incident on a surface , the flux density is called *IRRADIANCE* ($\boldsymbol{I \equiv \langle S \rangle}$).**

RADIATION PRESSURE

- If we consider a linearly polarised e-m wave traveling along the z direction , where the electric field is along the x-direction and the magnetic field is along the y-direction.
- The em wave interacts with the charge ‘q’ , such that the electric field exerts an upwards or a downwards force along the x-direction. This force is given by the Lorentz force as :

$$F = q\ (v\ x\ B)$$

- This force acts along the z-axis and constitutes the *RADIATION PRESSURE.*

$$F = qv\,B\ \ \check{z} \qquad ; \qquad B = \frac{E}{c}$$

$$\Rightarrow F = qv\frac{E}{c}\ \ \check{z}\ ;$$

where (qv E) is the work done by the em field on the charge per unit time

RADIATION MOMENTUM

Momentum per unit time is equal to the force, thus one gets:

$p = \frac{U}{c}\ \check{z}$, where u is the energy of the em wave.

Considering the special case of a PHOTON, where , $E = U = h\vartheta$:

$$p = \frac{(h\vartheta)}{c} \ \Rightarrow photon\ momrntum$$